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Bethe's Equation Is Incomplete for the XXZ Model at Roots of Unity

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We demonstrate for the six vertex and XXZ model parameterized by $\Delta = -(q+q^{-1})/2 \neq \pm 1$ that when $q^{2N}=1$ for integer $N \geqslant 2$ the Bethe's ansatz equations determine only the eigenvectors which are the highest weights of the infinite dimensional sl_2 loop algebra symmetry group of the model. Therefore in this case the Bethe's ansatz equations are incomplete and further conditions need to be imposed in order to completely specify the wave function. We discuss how the evaluation parameters of the finite dimensional representations of the sl_2 loop algebra can be used to complete this specification.

KEY WORDS: Bethe's anszatz; loop algebra; quantum spin chains.

I. INTRODUCTION

The study of the eigenvectors and eigenvalues of the Hamiltonian of the XXZ chain with periodic boundary conditions specified by

$$H = -\frac{1}{2} \sum_{j=1}^{L} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z) \tag{1.1}$$

where σ_j^i is the i Pauli spin matrix at site j was initiated by Bethe[(1)] for the case $\Delta = \pm 1$ in 1931 and has been studied[(2–5)] for $\Delta \neq \pm 1$ since 1958. A major result of these studies is that the ground state eigenvalue for any

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value of S^z is determined by an appropriate solution of what is called "Bethe's equation"

$$\left(\frac{\sinh \frac{1}{2}(v_j+i\gamma)}{\sinh \frac{1}{2}(v_j-i\gamma)}\right)^L = \prod_{\substack{l=1 \\ l\neq j}}^{L/2-|S^z|} \frac{\sinh \frac{1}{2}(v_j-v_l+2i\gamma)}{\sinh \frac{1}{2}(v_j-v_l-2i\gamma)} \tag{1.2}$$

where we use

$$-\Delta=\cos\gamma=\tfrac{1}{2}(q+q^{-1}), \qquad 0\leqslant\gamma\leqslant\pi \tag{1.3}$$

Here

$$S^z=\tfrac{1}{2}\sum_{j=1}^{L} \sigma_j^z \tag{1.4}$$

is a conserved quantum number since the operator in the right hand side commutes with the Hamiltonian (1.1) The eigenvalues of (1.1) are

$$E=-\frac{\Delta}{2}L-2\sum_{j=1}^{L/2-|S^z|}(\cos p_j-\Delta) \tag{1.5}$$

where

$$\cos p=-\cos\gamma+\frac{\sin^2\gamma}{\cosh v-\cos\gamma} \tag{1.6}$$

The corresponding momentum P is obtained from

$$e^{iP}=\prod_{j=1}^{L/2-|S^z|}\frac{\sinh \frac{1}{2}(i\gamma+v_j)}{\sinh \frac{1}{2}(i\gamma-v_j)} \tag{1.7}$$

There are many solutions of (1.2) but it is a simple matter[5] to determine the particular solution that leads to the ground state. However the derivation of the equations is not restricted to the ground state. Thus there arises the question of whether or not the totality of solutions of the Bethe's equation (1.2) will give all eigenvalues of (1.1). This is referred to as the completeness problem for the Bethe ansatz equation.

There is a large literature concerning this completeness problem[7–20] and the closely related problems of classifying solutions of Bethe's equation in terms of the string hypothesis[21–23] and quartets, and wide and narrow pairs.[24–26, 12, 13] As recently as ref. 20 it was stated that "it still remains to be settled whether the Bethe ansatz produces the complete set of eigenstates."

However it is not always straightforward to interpret precisely what has been done. For example in the special case of the XXX model ($\Delta=-1$)

it is shown in refs. 1, 8–10 that the $sl(2)$ symmetry of the Hamiltonian groups states in degenerate multiplets where, denoting by $S^z_{\max}$ the maximum value of S^z in the multiplet, there are $2S^z_{\max}+1$ states in the multiplet with the same energy and momentum and with the values $S^z=S^z_{\max}$, $S^z_{\max}-1, \ldots, -S^z_{\max}$. These states are all given by Bethe's equation as long as multiple occupancy of the states with $v_j=\pm\infty$ is allowed. On the other hand as is pointed out in ref. 15 these computations are based on the string hypothesis of ref. 21 and in refs. 25, 22, 27 it is demonstrated that there are states of the XXX model where this hypothesis fails.

Another example is the study of completeness made in ref. 14 where it is stated that when $|\Delta|<1$ the root of unity case

$$q^{2N}=1 \tag{1.8}$$

must be explicitly excluded for the completeness proof to hold because of the occurrence of additional degeneracies. However, in ref. 17 this root of unity case is precisely the case for which it is claimed that a combinatorial completeness is proven on the basis of assuming a counting given by the string hypothesis and in ref. 17 there is no explicit mention made of degenerate multiplets mentioned in ref. 14.

The study of the degeneracies of the XXZ model at roots of unity originates in the much more general studies of Baxter[28–30] on the XYZ model and has been considered in the more special case of XXZ by several authors.[31–36] Recently[37] it was shown in the root of unity case (1.8) that the Hamiltonian (1.1) and the transfer matrix of the related six vertex model[38–44] have the infinite dimensional sl_2 loop algebra as a symmetry. This infinite dimensional symmetry algebra groups eigenvalues into degenerate multiplets whose consecutive values of S^z differ by N and as an example when $S^z\equiv 0 \pmod N$ the state with $S^z=S^z_{\max}-lN$ has the binomial multiplicity $\binom{2S^z_{\max}/N}{l}$. This sl_2 loop algebra symmetry must also lead to consequences for the solutions of the corresponding Bethe's equation (1.2) but unlike the XXX model with the finite symmetry algebra of $sl(2)$ multiple occupancy of $v_j=\pm\infty$ does not explain this degeneracy. It is the primary purpose of this paper to study the relation of the sl_2 loop algebra symmetry discovered in ref. 37 to the Bethe's equation (1.2).

In order to efficiently study all the solutions of Bethe's equation we find it most useful to recall that (1.2) arises in Baxter's functional equation solution[45–46, 28–30] to the six vertex model where the (suitably normalized) transfer matrix $T(v)$ of the six vertex model satisfies

$$(-1)^{L/2-|S^z|}\, T(v)\, Q(v)=\sinh^L \tfrac{1}{2}(v-i\gamma)\, Q(v+2i\gamma)+\sinh^L \tfrac{1}{2}(v+i\gamma)\, Q(v-2i\gamma) \tag{1.9}$$

where $T(v)$ is defined in terms of the Boltzmann weights $W(\mu, \nu)|_{\alpha,\beta}$ with $\mu, \nu, \alpha, \beta = \pm 1$ as

$$T(v) = \mathrm{Tr}\, W(\mu_1, \nu_1)\, W(\mu_2, \nu_2) \cdots W(\mu_L, \nu_L) \tag{1.10}$$

where

$$\begin{aligned} W(1, 1)|_{1,1} &= W(-1, -1)|_{-1,-1} = \sinh \tfrac{1}{2}(v + i\gamma) \\ W(-1, -1)|_{1,1} &= W(1, 1)|_{-1,-1} = \sinh \tfrac{1}{2}(v - i\gamma) \\ W(-1, 1)|_{1,-1} &= W(1, -1)|_{-1,1} = \sinh i\gamma \end{aligned} \tag{1.11}$$

and $Q(v)$ is the auxiliary matrix introduced by Baxter which satisfies $[T(v), Q(v)] = [Q(v), Q(v')] = 0$. From these commutation relations and the fundamental relation that $[T(v), T(v')] = 0$ it follows that the matrix equation (1.9) also holds for the eigenvalues of $T(v)$ and $Q(v)$. Therefore if we write the eigenvalues of $Q(v)$ in the product form

$$Q(v) = \prod_{j=1}^{L/2 - |S^z|} \sinh \tfrac{1}{2}(v - v_j) \tag{1.12}$$

we find from (1.9) that the zeros v_j are given by (1.2) as long as the simultaneous vanishing

$$Q(v_j) = Q(v_j + 2i\gamma) = Q(v_j - 2i\gamma) = 0 \tag{1.13}$$

does not occur.

In general this is all which is known. However, in the special case where $S^z = 0$ Baxter in Eq. (101) of ref. 28 gives the following explicit expression for the matrix $Q(v)$ which is valid for all γ

$$Q(v)|_{\alpha,\beta} = \rho \exp\left(\tfrac{1}{4} i(\pi - \gamma) \sum_{1 \leqslant J < K \leqslant L} (\alpha_J \beta_K - \alpha_K \beta_J) + \frac{1}{4}(v - i\pi) \sum_{J=1}^{L} \alpha_J \beta_J \right) \tag{1.14}$$

where $\alpha_j, \beta_j = \pm 1$ are the eigenvalues of σ_j^z at the site j, ρ is a suitable normalizing constant and the restriction $S^z = 0$ means that L is even and that

$$\alpha_1 + \cdots + \alpha_L = \beta_1 + \cdots + \beta_L = 0 \tag{1.15}$$

The functional equation (1.9) and the exact expression (1.14) for Q with $S^z = 0$ will be the basis for most of our studies of the solutions v_j of Bethe's equation (1.2).

In order to make our statements precise we will first summarize in Section 2 what is meant by the string hypothesis of refs. 21 and 23 and formulate procedures for numerically determining the solutions v_j of the Bethe's equation (1.2). In Section 3 we will study these numerical solutions of (1.2) for the two values $\Delta=0, -1/2$ and compare with the picture given by the string hypothesis[21, 23] and the counting of ref. 17. For the case $\Delta=0$ the complete solution was given long ago by Lieb, Schultz and Mattis.[47] We will see that the v_j which are obtained from the Bethe's ansatz equation (1.2) in the limit $\Delta \to 0$ and from the exact expression (1.14) for Q with $S^z=0$ do not agree with the corresponding v_j of ref. 47 whereas the state counting results of ref. 17 do agree with ref. 47. Similarly for $\Delta=-1/2$ the roots obtained from continuity and from the exact expression (1.14) for Q with $S^z=0$ are also different from the string structure posited by ref. 17.

It must be stressed, however, that even though the numerical results we find for v_j do not agree with the results of ref. 47 for $\Delta=0$ our results do not contradict this paper because the eigenvectors being discussed are linear combinations of the Bethe states with degenerate eigenvalues which we obtained by continuity. We explain this in detail in Section 4 and relate this phenomenon to the picture obtained from the evaluation representation decomposition of the sl_2 loop algebra. For the case $\Delta=0$ we explicitly compute these evaluation parameters using Jordan Wigner techniques. For both $\Delta=0, -1/2$ we present an empirical relation between the evaluation parameters computed as roots of the associated Drinfeld polynomial and some of the numerical solutions of Section 3.

We conclude by remarking that even though for the explicit roots of unity with $N=2, 3$ presented in Section 3 the phenomenon of quartets[24–26, 12, 13] does not happen we have in fact found many examples for $N \geqslant 4$ where quartet states are present. The existence of these non-string states has nothing to do with the degeneracies resulting from the sl_2 loop algebra of ref. 37 and therefore we have not extended our examples to cases with $N \geqslant 4$. However these quartets are most interesting in their own right and for that reason we will discuss them separately elsewhere.

II. FORMULATION

In this section we make precise what is meant by the string hypothesis and we outline the procedure we will use to numerically study the roots v_j of the Bethe's equation (1.2).

A. The String Hypothesis

The detailed study of the solutions of the Bethe's equation (1.2) was begun in 1972 in ref. 21 where it is hypothesized that in the limit $L \to \infty$ the roots v_j form complexes of n values where the imaginary parts are either

$$\Im v = \begin{cases} (n+1-2k)\,\gamma & \mod 2\pi \\ (n+1-2k)\,\gamma + \pi & \mod 2\pi \end{cases} \tag{2.1}$$

with $k = 1, 2, \ldots, n$. In other words as $L \to \infty$ the roots of (1.2) are of the form (Eq. (2.9) of ref. 21)

$$v_{j,k} = \begin{cases} v_j^r + (n+1-2k)\, i\gamma + O(\exp(-\delta L)) & \mod 2\pi \\ v_j^r + (n+1-2k)\, i\gamma + \pi i + O(\exp(-\delta L)) & \mod 2\pi \end{cases} \tag{2.2}$$

where $\delta > 0$, v_j^r is real and $k = 1, 2, \ldots, n$. The first type of solution is called an n string of positive parity (denoted as $(n+)$) and the second type of solution is called an n string of negative parity (denoted by $(n-)$).

The form of solution (2.2) is universally referred to as the string hypothesis although some authors include the term $O(\exp(-\delta L))$ only implicitly. It is mandatory that these terms are included for all $n \geqslant 2$ because if they are not the right hand side of (1.2) will contain explicit factors of zero or infinity.

The next question which arises is which values of n are allowed for a given value of γ. For an irrational value of γ/π an infinite number of values of n are allowed when $L \to \infty$. However for the values $\gamma = m\pi/N$ (when the root of unity condition $q^{2N} = 1$ holds) the maximum value of n determined by the formalism of Eqs. (2.12)–(2.14) of ref. 21 is $n_{\max} = N - 1$.

For the purposes of this paper we will restrict our attention to the special case $\gamma = \pi/N$. Then the formalism of ref. 21 gives the following allowed states of strings: $(1,+), (2,+), \ldots, (N-1,+)$ and $(1,-)$.

There are several assumptions which have been tacitly made in these derivations which should be made explicit.

First of all there is the assumption which is sometimes made that all the real parts v_j^r in (2.2) should be finite. However, examples of infinite roots are known for both the XXX and XXZ models (see for example ref. 18) and for the XXX model these infinite roots are connected with the $sl(2)$ multiplet structure of the degeneracies. These infinite roots mean only that the number of finite roots is less than the assumed value of $\frac{L}{2} - |S^z|$ which appears in (1.2). We do not make this assumption of finiteness of v_j^r

in our definition of string state and thus these "singular solutions" do not violate our form of the hypothesis.

There is a more serious problem, however, which first seems to have been made explicit in 1973 by Baxter (p. 54 of ref. 30) who realized that the picture given above for $\gamma = m\pi/N$ is not complete and that strings of length N must also be allowed. We call these strings complete N strings. There is no restriction on parity for these strings. Therefore in the special case $\gamma = \pi/N$ the meaning of the string hypothesis as taken from ref. 23 is that the form for the roots is (2.2) and the allowed string length are $(1,+)$, ..., $(N,+)$, $(1,-)$, $(N,-)$. We note, however, that implicitly certain complete N strings are contained in the combinatorial completeness arguments of ref. 17 because this counting contains composite states consisting of an $(N-1,+)$ string and a $(1,-)$ string and for $N \geqslant 3$ this composite state will in fact be a complete N string if the real parts of $(N-1,+)$ and $(1,-)$ are equal.

There is however, a very important feature of these complete N strings of ref. 30 which sets them apart from all other strings of the form (2.2). Namely, spacings of the N different imaginary parts of the complete N strings are exactly given by 2γ for finite L. This feature is not present for all of the other strings and we will thus refer to these N strings as "exact complete N strings."

From (1.5) we see that the contribution of an exact complete N string to the energy is zero (independent of L). In other words the eigenstates of the Hamiltonian which differ only by exact complete N strings are degenerate in energy. This is exactly the degeneracy which arises from the sl_2 loop symmetry algebra found in ref. 37. From (1.7) we find the contribution to the momentum of the exact complete N string with $\gamma = \pi m/N$

$$P = \begin{cases} 0 & \text{if} \quad N-m \text{ is even} \\ \pi & \text{if} \quad N-m \text{ is odd} \end{cases} \tag{2.3}$$

The feature of the exact spacing of the imaginary parts of the roots of the exact complete N strings has the dramatic effect that there are now terms in the Bethe's equation (1.2) which are of the form 0/0 and therefore there is no equation left to determine the real part. Moreover, there is no equation to guarantee that v_j^r need be real if there is a state which contains two or more exact complete N strings.

B. Exact Complete *N* Strings in Baxter's *Q*

The meaning and the necessity of exact complete N strings comes from the sl_2 loop algebra symmetry and is easily seen when Baxter's matrix functional equation (1.9) is written in terms of the eigenvalues of $T(v)$ and $Q(v)$. When the root of unity condition $q^{2N}=1$ holds the transfer matrix has degenerate eigenvalues. But on the other hand the matrix $Q(v)$ does not have degenerate eigenvalues. Therefore the only way for the functional equation (1.9) to hold for the degenerate eigenvalues of $T(v)$ with several distinct polynomials $Q(v)$ is for $Q(v)$ to contain factors of the exact complete N string

$$Q_N(v)=\prod_{j=1}^{N}\sinh\tfrac{1}{2}(v-\alpha-2ji\gamma) \tag{2.4}$$

These factors obey $Q_N(v)=Q_N(v+2i\gamma)$ and hence the simultaneous vanishing condition (1.13). Therefore we conclude when the root of unity condition (1.8) holds that the functional equation (1.9) is not sufficient to determine the parameter α in the exact complete N strings (2.4) which exist if L is sufficiently large. Indeed the functional equation does not by itself even guarantee that the imaginary part of α is either 0 or π.

C. Solution by Continuity

For any fixed L any deviation of q from $q^{2N}=1$ will break all the degeneracies of the eigenvalues of $T(v)$ and now there will be a one-to-one relation between the eigenvalues of $T(v)$ and $Q(v)$. Therefore one way to determine the values of α in the exact complete N string of (2.4) is by continuity from the nondegenerate case. This will give a limiting set of solutions of the Bethe's ansatz equation at the roots of unity.

In principle for any given root of unity we can analytically determine a set of limiting Bethe's ansatz equations by continuity from (1.2) which resolves the ambiguity of 0/0. However, here we will follow an alternative numerical procedure which begins with the functional equation (1.9). Our procedure is as follows. Because of the commutation relation $[T(v),T(v')]=0$ the eigenvectors of $T(v)$ are independent of v and we may determine them numerically on the computer by choosing any convenient value of v we please. By letting $T(v)$ act on these v independent numerical vectors we may determine the eigenvalues as polynomials in e^v of degree L on the computer. We then determine the coefficients of the $\frac{L}{2}-|S^z|$ degree polynomial for $Q(v)$ by considering the functional equation (1.9) at $\frac{L}{2}-|S^z|+1$ different values of the spectral parameter v and solving the resulting system of homogeneous linear equations on the computer. As long as the eigenvalues of $T(v)$ are non degenerate this procedure is unique and unambiguous. The zeroes v_j of the eigenvalues of $Q(v)$ are then easily found by finding the roots of the $\frac{L}{2}-|S^z|$ order polynomials on the computer. The limit of $\Delta\to 0$ and $\Delta\to \pm 1/2$ is then obtained by studying sequences of Δ which approach the root of unity under consideration. Of course in practice there is an optimum value of Δ such that if we come closer to the root of unity than this value the accuracy of the computation will deteriorate. Fortunately for the cases considered in this paper this necessary limitation does not interfere with our ability to see the qualitative features of the limiting case.

This method of continuity can be done for any value of S^z. The validity of this approach to the limit is then checked for the special case $S^z=0$ by numerically diagonalizing the matrix (1.14) exactly at $\Delta=0$ ($\gamma=\pi/2$) and $\Delta=-\frac{1}{2}$ ($\gamma=\pi/3$).

III. NUMERICAL STUDIES FOR $L=16$

In this section we use the procedure outlined above to study the continuous solution to Bethe's equation (1.2) at $\Delta=0$ and $\Delta=-1/2$. The two cases are presented separately.

A. $\Delta=0$ ($\gamma=\pi/2$)

We have obtained the Bethe's roots for all 2^{16} eigenvalues of the $L=16$ chain from the Q of (1.14) at exactly $\Delta=0$ for $S^z=0$ and by the limiting procedure described above for all other values of S^z. The root content of all eigenvalues without exception is described by $(1,+)$, $(1,-)$ strings, exact complete 2 strings and infinite roots. The ground state contains only $(1,+)$ strings and its root content is given in Table 1.

The states are grouped into degenerate multiplets which have a highest weight (in terms of S^z) which have only $(1,+)$ and $(1,-)$ roots in the

Table 1. Root Content for $\Delta=0$ of the Ground State with $E=-10.251\cdots$ and $P=0$

−2.317785
−1.192878
−0.626402
−0.197623
0.197623
0.626402
1.192878
2.317786

Table 2. Types of Degeneracies of the Transfer Matrix $T(u)$ for the Case $\Delta = 0$ and $L = 16$

Maximum $S^z = 8$, one type		
$S^z = 8$	multiplicity $= 1$	
6	8	one 2 string
4	28	two 2 strings
2	56	three 2 string
0	70	four 2 strings
-2	56	three 2 strings
-4	28	two 2 strings
-6	8	one 2 string
-8	1	
Maximum $S^z = 7$, one type		
$S^z = 7$	multiplicity $= 1$	
5	7	one 2 string
3	21	two 2 strings
1	35	three 2 strings
-1	21	two infinite roots, two 2 strings
-3	7	2 infinite roots, one 2 string
-5	1	2 infinite roots
Maximum $S^z = 6$, one type		
$S^z = 6$	multiplicity $= 1$	
4	6	one 2 string
2	15	two 2 strings
0	20	three 2 strings
-2	15	two 2 strings
-4	6	one 2 string
-6	1	
Maximum $S^z = 5$ two types		
Type 1		
$S^z = 5$	multiplicity $= 1$	
3	4	one 2 string
1	6	two 2 strings
-1	4	two infinite roots, 2 strings
-3	1	two infinite roots, one 2 string
Type 2		
$S^z = 5$	multiplicity $= 1$	one infinite root
3	5	one infinite root, one 2 string
1	10	one infinite root, two 2 strings
-1	10	one infinite root, two 2 strings
-3	5	one infinite root, one 2 string
-5	1	one infinite root

Table 2. (Continued)

Maximum $S^z=4$ one type		
$S^z=4$	multiplicity $=1$	
2	4	one 2 string
0	6	two 2 strings
-2	4	one 2 string
-4	1	
Maximum $S^z=3$, two types		
$S^z=3$	multiplicity $=1$	
1	2	one 2 string
-1	1	two infinite roots
$S^z=3$	multiplicity $=1$	one infinite root,
1	3	one infinite root, one 2 string
-1	3	one infinite root, one 2 string
-3	1	one infinite root
Maximum $S^z=2$		
$S^z=2$	multiplicity $=1$	
0	2	one 2 string
-2	1	
Maximum $S^z=1$ two types		
$S^z=1$	multiplicity $=1$	(nondegenerate)
$S^z=1$	multiplicity $=1$	one infinite root
$S^z=-1$	multiplicity $=1$	one infinite root
Maximum $S^z=0$ one type		
$S^z=0$	multiplicity $=1$	(nondegenerate)

sector $S^z=0 \pmod 2$ but which may also contain an infinite root for $S^z=1 \pmod 2$. The highest weight states are not degenerate and are therefore identical with the states of ref. 47.

The remaining members of the multiplet contain exact complete 2 strings. The content of exact complete 2 strings and large roots for each type of multiplet is given in Table 2.

In Table 3 we give examples of multiplets with one exact complete 2 string.

Table 3. Examples of Multiplets with $S^z = 2, 0, -2$ in $L = 16$ and $\Delta = 0$ with One Exact Complete 2 String

$E = -8.750\cdots$			
$S^z = 2,\ P = 18\pi/16$			
-1.192796			
-0.626362			
-0.197613			
0.197606			
0.626353			
2.317550			
$S^z = 0,\ P = 2\pi/16$			
-1.192878		-1.192878	
-0.626402		-0.626402	
-0.197623		-0.197623	
0.197623		0.197623	
0.626402		0.626402	
2.317786		2.317786	
-0.562453	$+i\pi/2$	-0.562453	
-0.562453	$-i\pi/2$	-0.562453	$+i\pi$
$E = -7.249\cdots$			
$S^z = \pm 2,\ P = 18\pi/16$			
-2.317555			
-0.626359			
-0.197613			
0.197602			
0.626344			
2.317902	$+i\pi$		
$S^z = 0,\ P = 2\pi/16$			
-2.317786		-2.317786	
-0.626402		-0.626402	
-0.197623		-0.197623	
0.197623		0.197623	
0.626402		0.626402	
2.317786	$+i\pi$	2.317786	$+i\pi$
0.0		0.0	$+i\pi/2$
0.0	$+i\pi$	0.0	$-i\pi/2$

In Table 4 we consider multiplets with multiple exact complete 2 strings by considering the states whose highest weight is $S^z=6$ with momentum $2\pi/16$ in $S^z=0, \pm 4$ and $18\pi/16$ in $S^z=\pm 2, \pm 6$. There are seven such multiplets with energies 0, $\pm .2986...$, $\pm .72111...$, $\pm .55197....$ These multiplets have 6 states in $S^z=\pm 4$, 15 states in $S^z=\pm 2$ and 20 states in $S^z=0$. In Table 4 we give the root content of the state with $E=-.55197$ in $S^z=0$ and $S^z=2$ as an illustration. In particular we note that the following types of imaginary parts occur

1. 0 and π
2. $\pm \pi/2$
3. Pairs of two strings with imaginary parts other than 0, $\pm \pi/2$ and π.

In Fig. 1 we extend this by plotting the location of all Bethe's roots of all eigenvalues in the sector $S^z=0$. In this plot all roots whose imaginary part is not 0 or π come from exact complete 2 strings.

In Table 5 we give examples of multiplets with $S^z=3, 1, -1$ and $S^z=3, 1, -1, -3$.

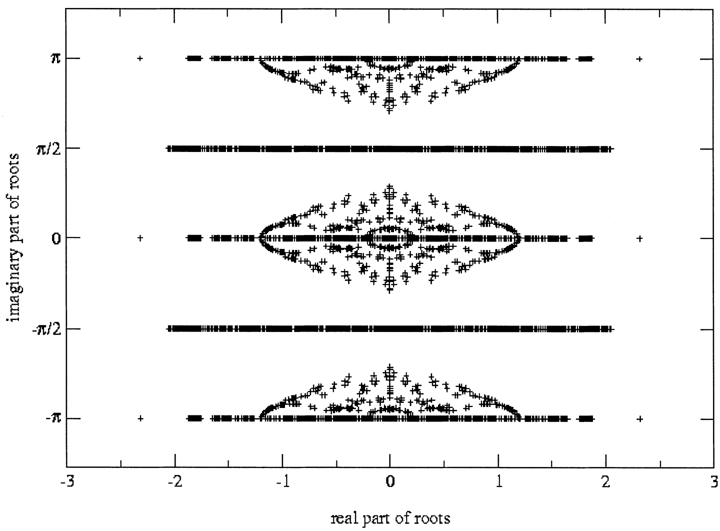

Fig. 1. Plot of the Bethe's roots for $\Delta=0$, $S^z=0$ as obtained from Baxter's exact expression for $Q(v)$. The roots are symmetric both about the real axis and under the reflection $v \to -v$.

Table 4. An Example of the Roots of a Degenerate Multiplet $S^z = 6, 4, 2, 0, -2, -4, -6$ in $\Delta = 0$, $L = 16$ with Highest Weight $S^z = 6$. We Choose the State with $E = -0.5517987$..... The Multiplet Has $2^6 = 64$ States. In $S^z = 0$ There Are 20 States, the Momentum Is $P = 2\pi/16$ and the Roots Are Computed from the Explicit Expression for Q (1.14). In $S^z = 2$ There Are 15 States, the Momentum Is $P = 18\pi/16$. All States Have the Same (1,+) and (1,−) Roots in Addition to Exact Complete 2 Strings.

$S^z=0$, $P=2\pi/16$			
$\sum \Im v_j \equiv 0 \pmod{2\pi}$		$\sum \Im v_j \equiv \pi \pmod{2\pi}$	
0.626402		0.626402	
1.192878	$+i\pi$	1.192878	$+i\pi$
−1.994484	$+i\pi/2$	−2.006672	$+i\pi/2$
−1.994484	$-i\pi/2$	−2.006675	$-i\pi/2$
−0.354311	$+i\pi/2$	−0.409599	$+i\pi/2$
−0.354311	$-i\pi/2$	−0.409599	$-i\pi/2$
1.439155		1.506631	$+i\pi/2$
1.439155	$+i\pi$	1.506631	$-i\pi/2$
0.626402		0.626402	
1.192878	$+i\pi$	1.192878	$+i\pi$
−1.877737	$+i\pi/2$	−1.862425	$+i\pi/2$
−1.877737	$-i\pi/2$	−1.862425	$-i\pi/2$
0.044698	$+i\pi$	0.067738	
0.044698		0.067738	$+i\pi$
0.923398	$+i\pi/2$	0.885045	
0.923398	$-i\pi/2$	0.885045	$+i\pi$
0.626402		0.626402	
1.192878	$+i\pi$	1.192878	$+i\pi$
−1.796958	$+i\pi/2$	−1.788035	$+i\pi/2$
−1.796958	$-i\pi/2$	−1.788035	$-i\pi/2$
−0.415405	$+i\pi$	−0.409732	
−0.415405		−0.409732	$+i\pi$
1.302723	$+i\pi/2$	1.288127	
1.302723	$-i\pi/2$	1.288127	$+i\pi$
0.626402		0.626402	
1.192878	$+i\pi$	1.192878	$+i\pi$
−1.852568	$+i\pi$	−1.847467	
−1.852568		−1.847467	$+i\pi$
−0.538335	$+i\pi/2$	−0.482661	$+i\pi/2$
−0.538335	$-i\pi/2$	−0.482661	$-i\pi/2$
1.481262	$+i\pi/2$	1.420488	
1.481262	$-i\pi/2$	1.420488	$+i\pi$

Table 4. (Continued)

$S^z=0,\ P=2\pi/16$			
$\sum \Im v_j \equiv 0 \pmod{2\pi}$		$\sum \Im v_j \equiv \pi \pmod{2\pi}$	
0.626402		0.626402	
1.192878	$+i\pi$	1.192878	$+i\pi$
−1.765722		−1.768190	
−1.765722	$+i\pi$	−1.768190	$+i\pi$
−0.421083		−0.426754	
−0.421083	$+i\pi$	−0.426754	$+i\pi$
1.277164		1.285303	$+i\pi/2$
1.277165	$+i\pi$	1.285303	$-i\pi/2$
0.626402		0.626402	
1.192878	$+i\pi$	1.192878	$+i\pi$
−1.793415		−1.799117	
−1.793415	$+i\pi$	−1.799117	$+i\pi$
0.067785		0.040394	
0.067785	$+i\pi$	0.040394	$+i\pi$
0.815989		0.849081	$+i\pi/2$
0.815989	$+i\pi$	0.849081	$-i\pi/2$
0.626402		0.626402	
1.192878	$+i\pi$	1.192878	$+i\pi$
−1.342393	$+i\pi/2$	−1.307213	$+i\pi/2$
−1.342393	$-i\pi/2$	−1.307213	$-i\pi/2$
−0.964462		−0.960063	
−0.964462	$+i\pi$	−0.960063	$+i\pi$
1.397214	$+i\pi/2$	1.357636	
1.397214	$-i\pi/2$	1.357636	$+i\pi$
0.626402		0.626402	
1.192878	$+i\pi$	1.192878	$+i\pi$
−1.122273	$+0.163271i$	−1.137586	$+0.146097i$
−1.122273	$-2.978321i$	−1.137586	$-2.995495i$
−1.122273	$-0.163271i$	−1.137586	$-0.146097i$
−1.122273	$+2.978321i$	−1.137586	$+2.995495i$
1.334906		1.365531	$+i\pi/2$
1.334906	$+i\pi$	1.365531	$-i\pi/2$
0.626402		0.626402	
1.192878	$+i\pi$	1.192878	$+i\pi$
−0.384740		−0.152129	$+i\pi/2$
−0.384740	$-i\pi$	−0.152129	$-i\pi/2$
−0.262449	$-2.561390i$	−0.378755	$-2.967736i$
−0.262449	$-0.580202i$	−0.378755	$-0.173856i$
−0.262449	$+2.561390i$	−0.378755	$+2.967736i$
−0.262449	$+0.580202i$	−0.378755	$+0.173856i$

Table 4. (Continued)

$\sum \Im v_j \equiv 0 \pmod{2\pi}$		$\sum \Im v_j \equiv \pi \pmod{2\pi}$	
$S^z=0$, $P=2\pi/16$			
0.626402		0.626402	
1.192878	$+i\pi$	1.192878	$+i\pi$
−0.805653		−0.857340	
−0.805653	$+i\pi$	−0.857340	$+i\pi$
−0.051993	$-2.957473i$	0.000361	
−0.051993	$+0.184119i$	0.000361	$+i\pi$
−0.051993	$+2.957473i$	−0.052661	$+i\pi/2$
−0.051993	$-0.184119i$	−0.052661	$-i\pi/2$
$S^z=2$, $P=18\pi/16$			
0.626343		0.626355	
1.192856	$+i\pi$	1.192850	$+i\pi$
−0.458130	$+i\pi/2$	−0.447383	$+2.958193i$
−0.458130	$-i\pi/2$	−0.447383	$-2.958193i$
−0.419474		−0.447366	$-0.183219i$
−0.419450	$+i\pi$	−0.447366	$+0.183219i$
0.626329		0.626368	
1.192864	$+i\pi$	1.192846	$+i\pi$
0.046111	$+i\pi/2$	0.004442	
0.046111	$-i\pi/2$	0.004435	$+i\pi$
−0.954798		−0.909222	
−0.954712	$+i\pi$	−0.909190	$+i\pi$
0.626364			
1.192848	$+i\pi$		
0.028744			
0.028697	$+i\pi$		
−0.883297	$+i\pi/2$		
−0.883297	$-i\pi/2$		
0.626287		0.626279	
1.192971	$+i\pi$	1.192969	$+i\pi$
−1.066835	$+i\pi/2$	−0.937278	
−1.066835	$-i\pi/2$	−0.937214	$+i\pi$
1.983971		1.981627	
1.984254	$+i\pi$	1.981907	$+i\pi$

Table 4. (Continued)

$S^z=2,\ P=18\pi/16$	
$\sum \Im v_j \equiv 0 \pmod{2\pi}$	$\sum \Im v_j \equiv \pi \pmod{2\pi}$
	0.626286 1.192966 $+i\pi$ −0.375322 −0.375280 $+i\pi$ 1.978775 1.979051 $+i\pi$
	0.626307 1.192960 $+i\pi$ 0.080561 0.080711 $+i\pi$ 1.972039 1.972307 $+i\pi$
0.626323 1.192867 $+i\pi$ 0.198855 $+i\pi/2$ 0.198855 $-i\pi/2$ −2.040445 −2.040077 $+i\pi$	
	0.626277 1.192970 $+i\pi$ 1.982814 1.983095 $+i\pi$ −2.037150 −2.036788 $+i\pi$
	0.626349 1.192853 $+i\pi$ −0.462058 −0.461966 $+i\pi$ −2.023406 −2.023066 $+i\pi$
	0.626367 1.192847 $+i\pi$ 0.017464 0.017436 $+i\pi$ −2.032311 −2.031957 $+i\pi$

Table 4. (Continued)

$S^z=2,\ P=18\pi/16$	
$\sum \Im v_j \equiv 0 \pmod{2\pi}$	$\sum \Im v_j \equiv \pi \pmod{2\pi}$
	0.626319
	1.192870 $+i\pi$
	0.304376 $+i\pi/2$
	0.304376 $-i\pi/2$
	−1.151308 $+i\pi/2$
	−1.151308 $-i\pi/2$
	0.626344
	1.192855 $+i\pi$
	−1.061563
	−1.061281 $+i\pi$
	−1.970389
	−1.970124 $+i\pi$

With the data from Fig. 1 we may now examine in much greater detail the meaning of the term "Bethe's ansatz equation" which is commonly used to refer to Eq. (1.2). Consider first directly setting $\gamma=\pi/2$ in (1.2). We see that (1.2) reduces to

$$\left(\frac{\sinh\frac{1}{2}\left(v_j+\frac{\pi i}{2}\right)}{\sinh\frac{1}{2}\left(v_j-\frac{\pi i}{2}\right)}\right)^L=(-1)^{L/2-|S^z|-1} \tag{3.1}$$

and thus

$$\frac{\sinh\frac{1}{2}\left(v_j+\frac{\pi i}{2}\right)}{\sinh\frac{1}{2}\left(v_j-\frac{\pi i}{2}\right)}=\begin{cases} e^{2\pi i m/L} & \text{if } \frac{L}{2}-|S^z|\equiv 1 \pmod 2 \\ e^{2\pi i(m+1/2)/L} & \text{if } \frac{L}{2}-|S^z|\equiv 0 \pmod 2 \end{cases} \tag{3.2}$$

and from (1.6)

$$p_m=\begin{cases} \frac{2\pi i m}{L} & \text{if } \frac{L}{2}-|S^z|\equiv 1 \pmod 2 \\ \frac{2\pi i}{L}(m+1/2) & \text{if } \frac{L}{2}-|S^z|\equiv 0 \pmod 2 \end{cases} \tag{3.3}$$

Table 5a. An Example of a Multiplet for $\Delta = 0$ and $L = 16$ with $S^z = 3, 1, -1$

$E = -7.875...$			
$S^z = 3$, $P = 2\pi/16$			
0.881310			
0.403173			
0.0			
−0.403166			
−1.614741			
$S^z = 1$, $P = 18\pi/16$			
0.881326		0.881384	
0.403174		0.403901	
0.0		0.0	
−0.403184		−0.403208	
−1.614809		−1.614809	
0.366737	$+i\pi/2$	0.366743	
0.366737	$-i\pi/2$	0.366038	$+i\pi$
$S^z = -1$, $P = 2\pi/16$			
0.881310			
0.403173			
0.0			
−0.403166			
−1.614741			
∞			
$-\infty$			

Table 5b. An Example of a Multiplet for $\Delta = 0$ and $L = 16$ with $S^z = 3, 1, -1, -3$

$E = -5.875...$					
$S^z = \pm 3$, $P = 2\pi/16$					
1.614739					
0.881307					
0.403186					
−0.403383					
∞					
$S^z = \pm 1$, $P = 18\pi/16$					
1.614739		1.614736		1.614735	
0.881307		0.881300		0.881298	
0.403186		0.403168		0.403164	
−0.403383		−0.403156		−0.403175	
∞		∞		∞	
−0.258692		−1.260395		−1.323623	$+i\pi/2$
−0.258590	$+i\pi$	−1.260335	$+i\pi$	−1.323623	$-i\pi/2$

with $m=0, 1, ..., L-1$. The solutions to (3.2) all satisfy

$$\Im v_j = 0, \qquad \text{or} \qquad \pi \tag{3.4}$$

which obviously is in gross contradiction to the example of Table 2 and with the large mass of data summarized in Fig. 1. On the other hand the roots of (3.1) are precisely the roots of the string ansatz of ref. 21 and agree exactly with the computation of Lieb, Schultz and Mattis.[47]

This disagreement between the solution of Bethe's equation (1.2) for Δ taken continuously to zero and the solutions of (3.1) was noted in ref. 20 where the equation (3.1) is referred to as Bethe's equation and with this terminology the authors are able to say that Bethe's equation is complete at $\Delta=0$ but that there is a discontinuity at $\Delta=0$. However, as we emphasized in Section 1 the Bethe's equation (1.2) is only derivable under the assumption that the simultaneous vanishing (1.13) does not occur. It seems to us more appropriate to preserve the condition (1.13) whenever we refer to (1.2) as the "Bethe's equation." Since the equation (3.1) holds even for the cases where the simultaneous vanishing (1.13) occurs we would prefer to call it the Lieb–Schultz–Mattis equation after its original discoverers. With this terminology we reserve the name "Bethe's equation" at roots of unity as the equation satisfied by the roots obtained by continuity as $\Delta \to 0$ for $S^z \neq 0$ and by the roots of $Q(v)$ of (1.14) at Δ exactly zero for $S^z=0$. With this terminology Bethe's equation will be continuous at $\Delta=0$ by definition; however the explicit form of the equation when there are exact complete 2 strings in the state is not known. Regardless of terminology it is a fact that the roots of the eigenvalues of $Q(v)$ given by (1.14) are not all given by (3.1).

B. $\Delta = -1/2$ ($\gamma = \pi/3$)

If this lack of continuity happened only at $\Delta=0$ it would perhaps only be of semantic interest whether or not we call (1.2) without the condition (1.13) by the name of Bethe's equation. But the phenomenon which we just saw at $\Delta=0$ happens for all Δ obtained from the root of unity condition (1.8). To make this specific we here explicitly consider the case $\Delta = -1/2$.

The root contents of the highest weight state of each multiplet are now made up of $(1,+)$, $(2,+)$ and $(1,-)$ strings. In Table 6 we give the root content of the ground state for $\Delta = \pm 1/2$ which contains only $(1,+)$ roots and the excited state (in $P=4\pi/16$) which contains a single $(2,+)$ string. In Table 7 we list the exact complete 3 string and infinite root content of all multiplets. In Table 8 we give several examples of multiplets with one exact complete 3 string and in Table 9 we give an example of a multiplet with $S^z=6, 3, 0, -3, -6$. In Table 10 we give examples of multiplets with $S^z=5, 2, -1, -4$, $S^z=5, 2, -1$ and $S^z=4, 1, -2$.

Table 6. The Root Content and the Ground State and the State with One (2, +) String for $\Delta = \pm 1/2$ for $L = 16$ Computed from $Q(v)$

$\Delta = -1/2$ $(\gamma = \pi/3)$		
$P = 0$ $E = -12.08552\cdots$	$P = 4\pi/16$ $E = -10.137668\cdots$	
-1.563663	-1.225672	$-1.048691i$
-0.798288	-1.225672	$+1.048691i$
-0.419617	-0.401731	
-0.132015	-0.117889	
0.132015	0.144763	
0.418617	0.431224	
0.798288	0.812236	
1.563663	1.582741	

$\Delta = +1/2$ $(\gamma = 2\pi/3)$		
$P = 0$ $E = -8.8272\cdots$	$P = 4\pi/16$ $E = -7.8380\cdots$	
-3.020233	-2.986276	
-1.583084	-1.546300	
-0.833007	-0.789520	
-0.262945	-0.205871	
0.262945	0.349177	
0.833007	0.996326	
1.583084	2.091232	$+2.083219i$
3.020233	2.091232	$-2.083219i$

Table 7. Types of Degeneracies of the Transfer Matrix $T(u)$ and the Hamiltonian for $\Delta = -1/2$ $(\gamma = \pi/3)$ and $L = 16$

	Maximum $S^z = 8$, one type	
$S^z = 8$	multiplicity $= 1$	
5	4	one 3 string
2	6	two 3 strings
-1	4	one 3 string, 4 infinite roots
-4	1	four infinite roots
	Maximum $S^z = 7$, one type	
$S^z = 7$	multiplicity $= 1$	
4	4	one 3 string
1	6	two 3 strings
-2	4	one 3 string, two infinite roots
-5	1	two infinite roots

Table 7. (Continued)

Maximum $S^z=6$, one type		
$S^z=6$	multiplicity $=1$	
3	4	one 3 string
0	6	two 3 strings
-3	4	one 3 string
-6	1	
Maximum $S^z=5$, two types		
Type 1		
$S^z=5$	multiplicity $=1$	
2	2	one 3 string
-1	1	four infinite roots
Type 2		
$S^z=5$	multiplicity $=1$	one infinite root
2	3	one infinite root, one 3 string
-1	3	two infinite roots, one 3 string
-4	1	two infinite roots
Maximum $S^z=4$, one type		
$S^z=4$	multiplicity $=1$	
1	2	one 3 string
-2	1	two infinite roots
Maximum $S^z=3$, one type		
$S^z=3$	multiplicity $=1$	
0	2	one 3 string
-3	1	
Maximum $S^z=2$, two types		
Type 1		
$S^z=2$	multiplicity $=1$	(nondegenerate)
Type 2		
$S^z=2$	multiplicity $=1$	one infinite root
-1	1	two infinite roots
Maximum $S^z=1$, one type		
$S^z=1$	multiplicity $=1$	(nondegenerate)
Maximum $S^z=0$, one type		
$S^z=0$	multiplicity $=1$	(nondegenerate)

Table 8. Examples of States with One Exact Complete 3 String for $\Delta = -1/2$ ($\gamma = \pi/3$)

$E = -8.3926\cdots$

$S^z = 3,\ P = 2\pi/16$		$S^z = 0,\ P = 2\pi/16$: $\sum \Im v_j \equiv 0 \pmod{2\pi}$		$S^z = 0,\ P = 2\pi/16$: $\sum \Im v_j \equiv \pi \pmod{2\pi}$	
0.562469		0.563138		0.563138	
0.263924		0.264228		0.264228	
0.007100		0.007099		0.007099	
−0.248840		−0.249147		−0.249147	
−0.959569		−0.960808		−0.960808	
		0.125163		0.125163	$+i\pi$
		0.125163	$+i2\pi/3$	0.125163	$+i\pi/3$
		0.125163	$-i2\pi/3$	0.125163	$-i\pi/3$

$E = -3.1921\cdots$

$S^z = 3,\ P = 2\pi/16$		$S^z = 0,\ P = 2\pi/16$: $\sum \Im v_j \equiv 0 \pmod{2\pi}$		$S^z = 0,\ P = 2\pi/16$: $\sum \Im v_j \equiv \pi \pmod{2\pi}$	
1.795639		1.798356		1.798356	
0.570485		0.571105		0.571105	
0.288133		0.288419		0.288419	
−0.434281		−0.434765		−0.434765	
−0.517966	$+i\pi$	−0.518450	$+i\pi$	−0.518450	$+i\pi$
		−0.568222	$+i\pi$	−0.568222	
		−0.568222	$+i\pi/3$	−0.568222	$+i2\pi/3$
		−0.568222	$-i\pi/3$	−0.568222	$-i2\pi/3$

$E = -2.6958\cdots$

$S^z = 3,\ P = 2\pi/16$		$S^z = 0,\ P = 2\pi/16$: $\sum \Im v_j \equiv 0 \pmod{2\pi}$		$S^z = 0,\ P = 2\pi/16$: $\sum \Im v_j \equiv \pi \pmod{2\pi}$	
1.760626		1.763480		1.763480	
0.528642		0.529315		0.529315	
0.238033		0.238347		0.238347	
−0.481042	$+i\pi/3$	−0.481597	$+i\pi/3$	−0.481597	$+i\pi/3$
−0.481042	$-i\pi/3$	−0.481597	$-i\pi/3$	−0.481597	$-i\pi/3$
		−0.522649		−0.522649	$+i\pi$
		−0.522649	$+i2\pi/3$	−0.522649	$+i\pi/3$
		−0.522649	$-i2\pi/3$	−0.522649	$-i\pi/3$

Table 9. A Multiplet in $P=2\pi/16$ with $S^z=6, 3, 0, -3, -6$ for $L = 16$ and $\Delta = -1/2$ ($\gamma=\pi/3$). The Roots for $S^z=6$ and 3 Are Taken for $\Delta = -0.501$. The Roots for $S^z=0$ Are Obtained from $Q(v)$ of (1.14) at Exactly $\Delta = -1/2$. The Root Content of the Highest Weight Is (1+), (1−). The Energy is $E=3.7394\cdots$

	$S^z=\pm 6$		
1.957258			
−2.540234	$+i\pi$		
	$S^z=\pm 3$		
$\sum \Im v_j \equiv 0 \pmod{2\pi}$		$\sum \Im v_j \equiv \pi \pmod{2\pi}$	
1.958725		1.958567	
−2.536625	$+i\pi$	−2.536538	$+i\pi$
0.051352	$+i\pi$	0.058681	$+i2\pi/3$
0.051227	$+i\pi/3$	0.058681	$-i2\pi/3$
0.051227	$-i\pi/3$	0.058681	
1.958683			
−2.536888	$+i\pi$		
−0.898706	$+i\pi$		
−0.894563	$+i\pi/3$		
−0.894563	$-i\pi/3$		
1.959915			
−2.536608	$+i\pi$		
1.171936	$+i\pi$		
1.165390	$+i\pi/3$		
1.165390	$-i\pi/3$		
	$S^z=0$		
$\sum \Im v_j \equiv 0 \pmod{2\pi}$		$\sum \Im v_j \equiv \pi \pmod{2\pi}$	
1.961385		1.961385	
−2.532983	$+i\pi$	−2.532983	$+i\pi$
−0.714134	$+i\pi/3$	−0.705970	$+\pi/3$
−0.714134	$-i\pi/3$	−0.705970	$-\imath\pi/3$
−0.714134	$+i\pi$	−0.705970	$+i\pi$
0.904667		0.896503	$+i\pi/3$
0.904667	$+i2\pi/3$	0.896503	$-i\pi/3$
0.904667	$-i2\pi/3$	0.896503	$+i\pi$

Table 9. (Continued)

$S^z=0$			
$\sum \Im v_j \equiv 0 \pmod{2\pi}$		$\sum \Im v_j \equiv \pi \pmod{2\pi}$	
1.961385		1.961385	
−2.532983	$+i\pi$	−2.532983	$+i\pi$
−0.718967		−0.742981	
−0.718967	$+i2\pi/3$	−0.742981	$+i2\pi/3$
−0.718967	$-i2\pi/3$	−0.742981	$-i2\pi/3$
0.909499	$+i\pi/3$	0.933514	
0.909499	$-i\pi/3$	0.933514	$+i2\pi/3$
0.909499	$+i\pi$	0.933514	$-i2\pi/3$
1.961385		1.961385	
−2.532983	$+i\pi$	−2.532983	
0.067492	$+i\pi/3$	0.095266	$+0.757222i$
0.067492	$-i\pi/3$	0.095266	$-0.757222i$
0.067492	$+i\pi$	0.095266	$+1.337172i$
0.123040	$+i2\pi/3$	0.095266	$-1.337172i$
0.123040	$-i2\pi/3$	0.095266	$+2.851671i$
0.123040		0.095266	$-2.851617i$

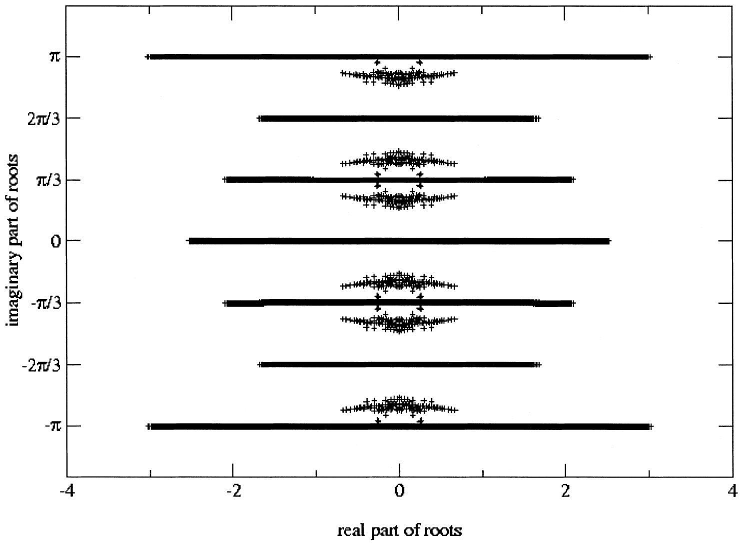

Fig. 2. Plot of the Bethe's roots for $\Delta=-1/2$, $S^z=0$ as obtained from Baxter's exact expression for $Q(v)$. The roots are symmetric both about the real axis and under the reflection $v \to -v$.

Table 10. Examples of Multiplets with $S^z_{max} = 5$ and 4 with $L = 16$ and $\Delta = -1/2$. The Roots Listed Are Taken from the Data for $\Delta = -0.501$ and We Indicate by $\pm\infty$ Roots Which Give Contributions to the Total Momentum of $\pm 2\pi/3$ in the Limit $\Delta \to -1/2$. We Give the Other Roots to 6 Places but Note that Some of the Values Can Differ from Their Values at $\Delta = -1/2$ by as Much as 0.07

a. An example of the multiplet with $S^z = 5, 2, -1, -4$					
$E = 4.057\cdots$, $P = 2\pi/16$					
$S^z = 5$					
-1.391351					
-0.304001	$+i\pi$				
$-\infty$					
$S^z = 2$					
-1.392407		-1.391696		-1.391722	
-0.305603	$+i\pi$	-0.302916	$+i\pi$	-0.302748	$+i\pi$
$-\infty$		$-\infty$		$-\infty$	
-0.979344	$+i\pi/3$	0.795702	$+i2\pi/3$	0.793797	$+i\pi/3$
-0.979344	$-i\pi/3$	0.795702	$-i2\pi/3$	0.793797	$-i\pi/3$
-0.984361	$+i\pi$	0.795701		0.794323	$+i\pi$
$S^z = -1$					
-1.392093		-1.391968		-1.392068	
-0.309847	$+i\pi$	-0.303059	$+i\pi$	-0.301622	$+i\pi$
∞		∞		∞	
∞		∞		∞	
-0.423815	$+i\pi/3$	-0.428616	$+i2\pi/3$	0.996443	$+i\pi/3$
-0.423815	$-i\pi/3$	-0.428616	$-i2\pi/3$	0.996443	$-i\pi/3$
-0.417366	$+i\pi$	-0.428616		1.000841	$+i\pi$
$S^z = -4$					
-1.391504					
-0.302815	$+i\pi$				
∞					
∞					

b. An example of the multiplet with $S^z = 5, 2, -1$

$E = -4.167\cdots$, $P = 2\pi/16$			
$S^z = 5$			
0.251789			
0.005266			
−0.519473			
$S^z = 2$			
0.251853		0.251846	
0.005442		0.005500	
−0.519607		−0.519683	
0.069044	$+i2\pi/3$	0.069117	$+i\pi/3$
0.069044	$-i2\pi/3$	0.069117	$-i\pi/3$
0.069101		0.06898	$+i\pi$
$S^z = -1$			
0.251816			
0.005524			
−0.519555			
∞			
∞			
$-\infty$			
$-\infty$			

c. An example of the multiplet with $S^z = 4, 1, -2$

$E = -5.478\cdots$, $P = 2\pi/16$			
$S^z = 4$			
0.721217			
0.128539			
−0.380667			
−0.706916			
$S^z = 1$			
0.721400		0.721478	
0.128630		0.128548	
−0.380765		−0.380829	
−0.707118		−0.707220	
0.072111		0.072073	$+i\pi$
0.072175	$+i2\pi/3$	0.072279	$+i\pi/3$
0.072175	$-i2\pi/3$	0.072279	$-i\pi/3$
$S^z = -2$			
0.721274			
0.128544			
−0.380688			
−0.706974			
∞			
$-\infty$			

In Fig. 2 we plot the position of the roots of all eigenvalues of $Q(v)$ in the sector $S^z=0$. We note as for $\Delta=0$ that the values of the imaginary parts which are not 0, $\pm\pi/3$ or π are all for roots of exact complete 3 strings.

IV. COMPLETENESS, INCOMPLETENESS AND EVALUATION PARAMETERS

As done above for $\Delta=0$ it is always possible to make the statement that Bethe's equation is complete in the degenerate cases if we can start from a completeness in the case where the parameter Δ is generic and then define the term "Bethe's equation" at the root of unity case by continuity. This, of course, has the serious disadvantage that at roots of unity most of the "Bethe's equations" are not yet known.

On the other hand if we define Bethe's equation to be (1.2) with the restriction (1.13) even at roots of unity (1.8) then the examples given above for $\Delta=-1/2$ demonstrate that Bethe's equation by itself is not complete. Instead with this definition Bethe's equation is complete for the highest weight of the multiplet and the remaining states must be obtained by applying appropriate lowering operators to this state of highest S^z.

To be more precise we turn to the theory of finite dimensional representations of (quantum) affine Lie algebras[48–50] where the states of a degenerate multiplet are specified by tensor products of evaluation representations. To determine the evaluation parameters of these representations (in the sector $S^z\equiv 0 \pmod N$) we use the two Chevalley generators of the sl_2 loop algebra $T^{+(N)}$ and $S^{-(N)}$ defined in ref. 37 to define the numbers μ_r from

$$\frac{T^{+(N)^r}}{r!}\frac{S^{-(N)^r}}{r!}\Omega=\mu_r\Omega \tag{4.1}$$

where Ω is the vector in the multiplet with the maximum value of S^z. From these μ_r we form the Drinfeld polynomial

$$P(x)=\sum_{r\geqslant 0}\mu_r(-x)^r \tag{4.2}$$

Then the evaluation parameters a_j are given as

$$P(x)=\prod_j (1-a_jx) \tag{4.3}$$

As a specific example Jimbo (private communication) has shown for a chain with $L \equiv 0 \pmod N$ with N odd and $q = e^{\pi i/N}$ that for the multiplet whose highest weight state is the vector with all spins up we have

$$\mu_r = \frac{L!}{(rN)!\,(L-rN)!} \tag{4.4}$$

and thus the corresponding Drinfeld polynomial is

$$P(x) = \frac{1}{N} \sum_{k=0}^{N-1} (1 - e^{2\pi k/N} x^{1/N})^L \tag{4.5}$$

For a complete solution to the problem we need an efficient method of computing the evaluation parameters and an explicit construction of the eigenvectors of $T(v)$ in terms of these evaluations parameters. Both of these problems are open at the present.

Moreover the relation of the evaluation representation to the exact complete N strings is not known. We do know empirically for the multiplets $S^z = N, 0, -N$ (which have two states in $S^z = 0$ and are specified by two evaluation parameters a_1, a_2) that the real parts of the complete exact N strings of both the states in $S^z = 0$ are equal and are given by

$$\alpha = \frac{1}{2N} \ln a_1 a_2 \tag{4.6}$$

We also know for the multiplet $S^z = 2N, N, 0, -N, -2N$ that if we consider the states in $S^z = 0$ with two complete exact N strings which have the same real part then this real part is given in terms of the four evaluation parameters as

$$\alpha = \frac{1}{4N} \ln \alpha_1 \alpha_2 \alpha_3 \alpha_4 \tag{4.7}$$

But for $\Delta = 0$ it was possible to go further and produce the alternative equation (3.1) and this equation is well defined even for solutions where the simultaneous vanishing (1.13) occurs. This equation did not give the roots of $Q(v)$ but did correctly give all eigenvalues and eigenvectors of the Hamiltonian. It can also be determined that all evaluation parameters are of the form

$$a_j = \cot^2 \frac{1}{2}\left(p_j + \frac{\pi}{2}\right) \tag{4.8}$$

where p_j is given by (3.3).

It is thus natural to ask if a similar procedure can be done for other values of Δ. This seems to be the approach to completeness taken in ref. 17 where a string ansatz is made for the solutions of (1.2) whenever the equation is well defined and then (tacitly) applied for the cases where the vanishing condition (1.13) holds and the equations is not defined. This procedure gave the correct counting but because no equation comparable to (3.1) is given there it is not possible to construct the corresponding degenerate states in the multiplet. Furthermore the relation which this procedure has to the evaluation representation of the sl_2 loop algebra symmetry is not known.

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